Certain Results on the Conharmonic Curvature Tensor of $LAP-$Manifolds

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Abstract— **The object of the paper is to study Lorentzian almost para- contact manifolds (briefly LAP−manifolds) satisfying certain curva- ture coditions on conharmonic curvature tensor. In the present paper we discuss about conharmonically pseudosymmetric, partially Ricci- pseudosymmetric, conharmonically φ−symmetric and φ−conharmonically flat LAP−manifolds.Specially, we study LAP−manifold with conhar- monically flat curvature tensor which is locally isometric with unit sphere Sn(1)**

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I. Introduction

Paracontact geometry, in particular, is important due to its connection with the theory of para-Kahler manifolds and its relevance in pseudo-Riemannian geometry and mathematical physics. Recently, LAP geometry has emerged as an intriguing and vital area within differential geometry.

In 1989, K. Matsumoto introduced the concept of LP−Sasakian manifolds [85]. This concept was independently introduced by I. Mihai and R. Rosca [87]. Lorentzian para-Sasakian (LP-Sasakian) manifolds have been further explored by K. Matsumoto and I. Mihai [86], U. C. De and A. A. Shaikh [88], and many others ([13], [21], [93], [119], [123]). K. Matsumoto and T. Adati obtained notable results regarding conformally recurrent and conformally sym46 metric P-Sasakian manifolds [2]. The idea of a semi-symmetric connection on a differentiable manifold was first introduced by Friedmann and Schouten in 1924 [49]. In 2008, Venkatesha and C.S. Bagewadi [127] generalized the concept of locally concircular ϕ-symmetric LP-Sasakian manifolds by obtaining results on concircular ϕ-recurrent LP-Sasakian manifolds.

A transformation of an n−dimensional Riemannian manifold N that maps every geodesic circle in N to another geodesic circle is known as a concircular transformation ([78], [133]). Such a transformation is always conformal [78]. Significant interest attached to a special type of conformal transformations is known as conharmonic transformations (i.e., conformal transformations that keep the property of smooth harmonic functions). In 1957, Y. Ishii [71] introduced this type of transformation. It is easy to verify that this tensor is an algebraic curvature tensor which has classical symmetry properties of the Riemann curvature tensor. The completion of a Riemannian structure to almost Hermitian structures allow additional symmetry properties of the conharmonic curvature tensor.

A rank four tensor L that remains invariant under conharmonic transformation for a 2n + 1−dimensional Riemannian manifold M^{2n+1} , is given by[71]

$$\bar{L}(U,V,W,X) = \bar{R}(U,V,W,X) - \frac{1}{2n-1}[g(V,W)S(U,X) - g(U,W)S(V,X)] + S(V,W)g(U,X) - S(U,W)g(V,X),$$

where R denotes the Riemannian curvature tensor of type (0, 4) and L denotes the conharmonic curvature tensor of type (0, 4) defined by

$$\bar{R}(U,V,W,X) = g(R(U,V),W,X), \bar{L}(U,V,W,X) = g(\bar{L}(U,V)W,X)$$

(3.1.2)

Where R is the Riemannian curvature tensor of type (1, 3) and S denotes the Ricci tensor of type (0, 2). The Conharmonic curvature tensor has been studied by Abdussattar [1], Özgür [94], Siddiqui et al. [122], Ghosh et al. [51] and many others. We exhibit our work as follows: Section 2 contains brief account on LAP−manifolds which will be used later. In Section 3, we study conharmonically pseudosymmetric LAP−manifold and find some results. Section 4 deals with partially Ricci-pseudosymmetric LAP−manifolds. Section 5 is devoted to the study of conharmonically ϕ−symmetric LAP−manifolds. In the last section we discuss about ϕ−conharmonically flat LAP−manifold and prove that it is a generalized η−Einstein manifold and also locally isometric with unit sphere S n (1)

II. Preliminaries

An n-dimensional differentiable manifold N is said to be a Lorentzian almost para-contact manifold, if it admits an almost para-contact structure (ϕ, ξ, η, g) consisting of a (1, 1) tensor field ϕ, vector field ξ, 1-form η and a Lorentzian metric g satisfying

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, g(U,\xi) = \eta(U),$$
(3.2.1)

$$\phi^2 U = U + \eta(U)\xi,$$
(3.2.2)

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V),$$
(3.2.3)

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$$(\nabla_U \eta)V = g(U, \phi V) = (\nabla_V \eta)U,$$
(3.2.4)

for any vector field U, V on M. Such a manifold N is termed as Lorentzian para-contact manifold and the structure (ϕ, ξ, η, g) a Lorentzian para-contact structure [85].

Definition 2.1. A LAP manifold N is called Lorentzian para-Sasakian manifold or briefly LP−Sasakian manifold if (ϕ, ξ, η, g) satisfies the conditions

$$d\eta = 0, \qquad \nabla_U \xi = \phi U,$$
(3.2.5)

$$(\nabla_U \phi)V = g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi,$$
(3.2.6)

for U, V tangent to M, where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a LP−Sasakian manifold N with respect to the Levi-Civita connection ∇ satisfies the following relations [101]

$$\eta(R(U,V)W) = g(V,W)\eta(U) - g(U,W)\eta(V),$$
(3.2.7)

$$R(\xi,U)V = g(U,V)\xi - \eta(V)U,$$
(3.2.8)

$$R(\xi,U)\xi = -R(U,\xi)\xi = U + \eta(U)\xi$$
(3.2.9)

$$R(U,V)\xi = \eta(V)U - \eta(U)V,$$
(3.2.10)

$$S(U,\xi) = (n-1)\eta(U), \qquad Q\xi = (n-1)\xi$$
(3.2.11)

$$S(\phi U, \phi V) = S(U,V) + (n-1)\eta(U)\eta(V),$$
(3.2.12)

for all vector fields For all vector fields U,V,W $\epsilon\ \Gamma\ (TM)$

Definition 2.2. A LAP−Sasakian manifold $\mathcal{N}$ is said to be an η−Einstein manifold [101] if its Ricci tensor S of the Levi-Civita connection is of the form

$S(U,V) = ag(U,V) + b\eta(U)\eta(V)$ S (U, V) for all U, V ∈ Γ(T M) (3.2.13)

where a and b are smooth functions on the manifold $\mathcal{N}$

Definition 2.3. A LAP manifold $\mathcal{N}$ is called Lorentzian para-Kenmotsu manifold if (ϕ, ξ, η, g) satisfies the conditions

$$(\nabla_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi\,(U),$$
(3.2.14)

For any vector field U,V on M.

In the Lorentzian para-Kenmotsu manifold ,we have

$$\nabla_U \xi = -U - \eta(U)\xi,$$
(3.2.15)

$$(\nabla_U \eta)V = -g(U,V) - \eta(U)\eta(V),$$

Where ∇ the operator of covariant differentiation with respect to the Lorentzian metric g.

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Kenmotsu manifold N with respect to the Levi-Civita connection ∇ also satisfies the relations (3.2.7)-(3.2.12).

Example [126] Let M = {$(u^1, u^2, \dots, u^m, v^1, v^2, \dots, v^m$, w)} = $(u^i, v^i, w)\ \epsilon\ \mathbb{R}^{2m+1}$, where $(u^i, v^i, w\ \epsilon\ \mathbb{R}$ and i = 1,2,3,….,m) denote an n(= 2m + 1)- dimensional smooth manifold.

Let us define the structure tensor ϕ as:

$\phi(\xi) = 0, \phi(U_i) = V_i, \quad \phi V_i = U_i.$

If g represents Lorentzian metric of $\mathcal{N}$ defined by

$$g = -(\eta \otimes \eta) + e^{-2w} \sum_{i=1}^{n} (du^i \otimes du^i + (dv^i \otimes dv^i),$$

Then by linearity properties, we can easily show that the relations

$\phi^2 U = U + \eta(U)\xi, \qquad g(U,\xi) = \eta(U)$

Hold for all vector fields X on $\mathbb{R}^{2m+1}$. Thus (M,ϕ, ξ, η, g) forms a Lorentzian para-Kenmotsu manifold with respect to the $\phi - basis$ $U_i = e^w \frac{\partial}{\partial u^i}$, $Y_i = e^w \frac{\partial}{\partial v^i}$, and $\xi = \frac{\partial}{\partial w}$, where i = 1,2,…,m.

Definition 2.4. The concircular curvature tensor $\bar{L}$ of type (1,3) on LAP-manifold $\mathcal{N}$ of dimension n is defined by

$$\tilde{L}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]$$
(3.2.16)

For any vector fields X,Y,Z on $\mathcal{N}$ where R is the curvature tensor and r is the scalar curvature.

Definition 2.5. The conharmonic curvature tensor $\bar{L}$ of type (1,3) on LAP- manifold $\mathcal{N}$ of dimension n is defined by

$$\bar{L}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y]$$
(3.2.17)

For all vectors fields X,Y,Z on $\mathcal{N}$ where $\bar{Q}$ is the Ricci operator, i.e.

$g(\bar{Q}X, Y) = S(X,Y).$

If $\bar{L}$ vanishes identically then we say that manifold is harmonically flat [42].

III. Conharmonically Pseudosymmetric LAP-Manifolds

A Riemannian manifold $\mathcal{N}$ is said to be conharmonically pseudosymmetric if

$$R.\bar{L} = A_{\bar{L}}\mathcal{B}(g,\bar{L})$$
(3.3.1)

Holds on the set $\lambda_{\bar{L}} = \{p \epsilon M : \bar{L} \neq 0\}$ at p, where $A_{\bar{L}}$ is some function on $\lambda_{\bar{L}}$, R is curvature tensor, $\bar{L}$ is conharmonic curvature tensor and $\mathcal{B}$ is (0, k + 2) tensor field ,Let an N-dimensional (n > 2) LP-sasakian manifold $\mathcal{N}$ be a conharmonically pseudosymmetric .Then from (3.3.1) we have for X,Y,Z on M,

$$(R(U,\xi).\bar{L})(X,Y)Z = A_{\bar{L}}\left[\left((U \wedge_g \xi).\bar{L}\right)(X,Y)Z\right], \quad (3.3.2)$$

Where X $\wedge_B$ Y is the endomorphism and defined as following for a symmetric (0,2)-tensor field B on M for U,V,W on M

$$(U \wedge_B V)W = B(V,W)U - B(U,W)V. \quad (3.3.3)$$

Now the left hand of (3.3.2) in view 0f (3.2.8) and right hand side of (3.3.2) in view of (3.3.3) reduces the (3.3.2) in following form

$$(1 - A_{\bar{L}})[g(\xi,\bar{L}(X,Y)Z)U - g(U,\bar{L}(X,Y)Z)\xi] -\eta(X)\bar{L}(U,Y)W + g(U,X)\bar{L}(\xi,Y)Z - \eta(Y)\bar{L}(X,U)Z$$

$$+g(U,Y)\bar{L}(X,\xi)Z - \eta(Z)\bar{L}(X,Y)U + g(U,Z)\bar{L}(X,Y)\xi = 0, \quad (3.3.4)$$

Which implies that either $A_{\bar{L}} = 1$ or

$$[g(\xi,\bar{L}(X,Y)Z)U - g(U,\bar{L}(X,Y)Z)\xi] -\eta(X)\bar{L}(U,Y)Z + g(U,X)\bar{L}(\xi,Y)Z - \eta(Y)\bar{L}(X,U)Z$$

$$+g(U,Y)\bar{L}(X,\xi)Z - \eta(Z)\bar{L}(X,Y)X + g(U,Z)\bar{L}(X,Y)\xi = 0, \quad (3.3.5)$$

Taking inner product of (3.3.5) with ξ and using (3.2.1) and (3.2.2), we get

$$0 = \eta\bar{L}(X,Y)Z)\eta(U) + \bar{L}(X,Y,Z,U) - \eta(X)\eta(\bar{L}(U,Y)Z)$$

$$-\eta(Y)\eta(\bar{L}(X,U)Z) - \eta(\bar{Z})\eta(\bar{L}(X,Y)U) \quad . \quad (3.3.6)$$

By simplifying we get

$$\bar{L}(X,Y,Z,U) = 0, \quad (3.3.7)$$

Which implies that M is conharmonically flat.
Therefore we can state that:

Theorem 3.1. If an n−dimensional (n > 2) LAP manifold $\mathcal{N}$ is conharmonically pseudosymmetric then $\mathcal{N}$ is either conharmonically flat or locally isometric to the unit sphere $S^n(1)$.

Corollary 3.2 If $\mathcal{A}_{\bar{L}}=0$ on $\lambda_{\bar{L}}$ then an n−dimensional (n > 2) LAP conharmonically pseudosymmetric manifold M is conharmonically semisymmetric then M is conharmonically semisymmetric.

Corollary 3.3. if an n-dimensional (n>2) Lorentzian almost paracontact manifold M is conharmonically semisymmetric then M is locally isometric to the unit sphere $S^n(1)$**.**

Since $\mathcal{A}_{\bar{L}}$ need not be zero in general so also there exists conharmonically pseudosymmetric manifolds which are not conharmonic semisymmetric. So if $\mathcal{A}_{\bar{L}} \neq 0$, it is easy to see that R.$\bar{L}$=$\mathcal{B}(g,\bar{L})$, which implies that the pseudosymmetric function $\mathcal{A}_{\bar{L}}=1$. Hence, we can state the following:

Theorem . Every LP−Sasakian manifold is conharmonically pseudosymmetric of the form R.$\bar{L}$=$\mathcal{B}(g,\bar{L})$.

3.4 Partially Ricci-pseudosymmetric LAP−manifolds

In this section we discuss the conditions about LP−Sasakian manifold (which is a type of LAP−manifold) for which it is to be partially Ricci-pseudosymmetric.

Definition 3.4.1. An n−dimensional (n > 2) LP−Sasakian manifold M is said to be partially Ricci-pseudosymmetric if

$$R.S = \psi(a)\mathcal{B}(g,S) \quad (3.4.1)$$

holds on the set $Q = \{p \in M : B(g,S) = 0 \text{ at } p\}$*, where* $\psi \in C\infty(Q)$ *for a* $\in Q$*. As we know that for all U,V,X and Y on M,*

$$(R(U,V).S)(X,Y) + S(R(U,V)X,Y) + S(X,R(U,V)Y) = 0, \quad (3.4.2)$$

And

$$\mathcal{B}(g,S) = ((U \wedge_g V.S)(X,Y), \quad (3.4.3)$$

The equation (3.4.1) can be written using (3.4.3) in following form

$$(R(U,V).S)(X,Y) = \psi(a)\left((U \wedge_g V).S\right)(X,Y), \quad (3.4.4)$$

Let an n-dimensional (n>2) LP-Sasakian manifold M is partially Ricci-pseudosymmetric, Then we can get from (3.3.3) and (3.4.4) for all U,V,X and Y on M,

$$S(R(U,V)X,Y) + S(X,R(U,V)Y) \quad (3.4.5)$$

$$= \psi(a)[S\left((U \wedge_g V)X,Y\right) + S(X,(U \wedge_g V(Y)].$$

Putting Y = V = ξ in above expression and using (3.2.8), (3.2.11), (3.3.3) and (3.4.5) , we get

$$(n-1)\{\eta(U)\eta(X) + g(U,X)\} - S(X,U) - (n-1)\eta(U)\eta(X) = \psi(a)[(n-1)\eta(U)\eta(X) + g(U,X) - \eta(U)\eta(X) - S(X,U)],$$

Hence get

$$[\psi(a) - 1][S(U,X) - (n-1)g(U,X)] = 0. \quad (3.4.6)$$

The equation (3.4.6) has two possibilities either

$\psi(a) = 1$ *or*

$S(U,X) = (n-1)g(U,X)$

The (2) shows that manifold is Einstein manifold.
Therefore we can state that:

Theorem3.4.1 . An n−dimensional (n > 2) partially Ricci-pseudosymmetric LP−Sasakian manifold M is an Einstein manifold iff $\psi(a) \neq 1$.

3.5 Conharmonically ϕ−symmetric LAP−manifolds

In this section we study about conharmonically ϕ−symmetric Lorentzian para-Kenmotsu manifold (which is a type of LAP−manifold) and find the interesting result.

Definition 3.5.1. A Lorentzian para-Kenmotsu manifold M is said to be conharmonically ϕ−symmetric if the conharmonic curvature tensor $\bar{L}$ satisfies

$$\phi^2\left((\nabla_U\bar{L})(X,Y)Z\right) = 0, \quad (3.5.1)$$

For all vector fields X,Y,Z and U on M.

Let a Lorentzian para-Kenmotsu manifold N of dimension n(> 2) be conharmonically ϕ−symmetric. Then the above relation (3.5.1) can be written as following in view of (3.2.1) and (3.2.2),

$$(\nabla_U\bar{L})(X,Y)Z + \eta\left((\nabla_U\bar{L})(X,Y)Z\right)\xi = 0,$$

(3.5.2)

Using (3.5.2) and (3.1.1) for dimension n, and putting X = V = e_i (where $\{e_i\}$, i = 1, 2, ..., n, is an orthonormal basis of the tangent space at each point of the manifold M) and taking summation over i, we get

$$\eta\left((\nabla_U R)(e_i,Y)Z\right)\eta(e_i) - \frac{1}{n-2}[g((\nabla_U\bar{Q})e_i,e_i \quad)$$

(3.5.3)

$$+((\nabla_U\bar{Q})e_i)\eta(e_i)]g(Y,Z)] + \frac{1}{n-2}[g((\nabla_U\bar{Q})Y,Z)$$

$$+(\nabla_U S)(\xi,Z)\eta(Y) + \eta((\nabla_U\bar{Q})Y)\eta(Z)] = 0.$$

Putting $Z = \xi$ in (3.5.3) , we get

$$\eta\left((\nabla_U R)(e_i,Y)\xi\right)\eta(e_i) - \frac{1}{n-2}[dr(U)\eta(Y)$$

(3.5.4)

$$+((\nabla_U\bar{Q})e_i)\eta(e_i) - (\nabla_U S)(\xi,\xi)] = 0$$

From (3.5.4) we get

$$\eta((\nabla_U\bar{Q})e_i)\eta(e_i) = g\left((\nabla_U\bar{Q})e_i,\xi\right)g(e_i,\xi) \quad (3.5.5)$$

$$= g\left((\nabla_U\bar{Q})\xi,\xi\right) = 0.$$

$$\eta\left((\nabla_U R)(e_i,Y)\xi\right)\eta(e_i) = 0,$$

(3.5.6)

And,

$$(\nabla_U S)(\xi,\xi) = 0$$

(3.5.7)

Using (3.5.5)-(3.5.7) and from (3.5.4) , we obtain

$$dr(Z) = 0$$

(3.5.8)

The above result (3.5.8) implies that r is constant. Therefore, we can state the following:

Theorem 3.5.1. If an n−dimensional Lorentzian para-Kenmotsu manifold M is conharmonically ϕ−symmetric then the scalar curvature r is constant.

3.6 ϕ− Conharmonically flat LAP−manifolds

In this section we study about ϕ− conharmonically flat Lorentzian para-Kenmotsu manifolds and also prove an interesting result on it. The notion of ϕ− conharmonically flat K−contact manifolds was first introduced by G. Zhen [135].

Analogous to the (3.2.17), the conharmonic curvature tensor $\bar{L}$ of type (1, 3) for X, Y, Z on is defined by

$$\bar{L}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y]$$

(3.6.1)

Definition 3.6.1. A Lorentzian para-Kenmotsu manifold of dimension n be a ϕ −conharmonically flat if

$$\phi^2\bar{L}(\phi X,\phi Y)\phi Z) = 0,$$

(3.6.2)

For any vector fields X,Y,Z on M.

Let a n-dimensional (n>2) lorentzian para-Kenmotsu manifold $\mathcal{N}$ be ϕ − conharmonically flat. Then from (3.6.2), it follows that

$$g(\bar{L}(\phi X,\phi Y)\phi Z,\phi W) = 0.$$

From (3.6.1) and (3.6.3), we have

$$g(R(\phi X,\phi Y)\phi Z,\phi W) = \frac{1}{n-2}[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W) + S(\phi X,\phi W)g(\phi Y,\phi Z) - S(\phi Y,\phi W)g(\phi X,\phi Z)]$$

(3.6.4)

Let $\{e_1, e_{2,\ldots\ldots,} e_{n-1}, \xi\}$ be a orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, \ldots\ldots, \phi e_{n-1}, \xi\}$ is also a orthonormal basis in M, if we put X = W = e_i in (6.4) and sum up with respect to I, we have

$$\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i)$$

$$= \frac{1}{n-2}\sum_{i=1}^{n-1}[S(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - S(\phi e_i,\phi Z)g(\phi Y,\phi e_i) + S(\phi e_i,\phi e_i)g(\phi Y,\phi Z) - S(\phi Y,\phi e_i)g(\phi e_i,\phi Z)]$$

(3.6.5)

It can be easily verify that

$$\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) = S(\phi Y,\phi Z) + g(\phi Y,\phi Z),$$

(3.6.6)

$$\sum_{i=1}^{n-1} S(\phi e_{i,}\phi e_i) = r + (n-1)$$

(3.6.7)

$$\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Z)S(\phi Y,\phi e_i) = S(\phi Y,\phi Z),$$

(3.6.8)

$$\sum_{i=1}^{n-1} g((\phi e_i,\phi e_i)) = (n+1),$$

(3.6.9)

So by using (3.6.6)-(3.6.9) the equation (3.6.5) will become

$$S(\phi Y,\phi Z) + g(\phi Y,\phi Z) = \frac{1}{n-2}[(n+1)S(\phi Y,\phi Z) - S \quad + (r+(n-1))S(\phi Y,\phi Z) - S(\phi Y,\phi Z)].$$

(3.6.10)

On solving we obtain

$$S(\phi Y,\phi Z) = -(r+1)g(\phi Y,\phi Z).$$

Using (3.2.3), (3.2.12) and (3.6.11), we get

$$S(Y,Z) = -(r+1)g(Y,Z) - (r+n)\eta(Y)\eta(Z),$$

On contracting, we get

$$r = 0.$$

Putting the value of (3.6.13) in (3.6.12), (3.6.12) turns into

$$S(Y,Z) = -g(Y,Z) - n\eta(Y)\eta(Z),$$
(3.6.14)

Which shows that $\mathcal{N}$ is an η −Einstein manifold, this leads us to state the following:

Theorem 3.6.1. An n−dimensional (n > 2) ϕ−conharmonically flat Lorentzian para-Kenmotsu manifold is a generalized η−Einstein manifold.

Theorem 3.6.2. A conharmonically flat Lorentzian para-Kenmotsu manifold N is locally isometric with the unit sphere S^n (1), where S is a Lorentzian manifold of sectional curvature one.

Proof. If $\bar{L} = 0$ then we get from (3.2.17) that

$$R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y].$$
(3.6.15)

Putting $Z = \xi$ in (3.6.15) and using (3.2.10) and (3.2.11) we obtain

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-2}[(n-1)\eta(Y)X - (n-1)\eta(X)Y + \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y].$$
(3.6.16)

Putting $Y = \xi$ in (3.6.16) and using (3.2.1) we get

$$-X - \eta(X)\xi = \frac{1}{n-2}[-(n-1)X-(n-1)\eta(X)\xi - \bar{Q}X + \eta(X)\xi],$$
(3.6.17) On simplifying (3.6.17), we get

$$\bar{Q}X = -X$$
(3.6.18)

Proceeding in the same way with $X = \xi$ in (3.6.16) we get

$$\bar{Q}Y = -Y$$
(3.6.19)

Using (3.6.18) and (3.6.19) in (3.6.15) we have

$$R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y - g(Y,Z)X + g(X,Z)Y],$$
(3.6.20)

Putting $X = \xi$ and using (3.2.8) and (3.2.11) we get

$$g(Y,Z)\xi - \eta(Z)Y = \frac{1}{n-2}[S(Y,Z)\xi - (n-1)\eta(Z)Y - g(Y,Z)\xi + \eta(Z)Y],$$ (3.6.21)

On simplification we get

$$S(Y,Z) = (n-1)g(Y,Z)$$
(3.6.22)

Therefore, manifold is an Einstein manifold.

Now, putting (3.6.18), (3.6.19) and (3.6.21) in (3.6.15) we have

$$R(X,Y)Z = \frac{1}{n-2}[(n-1)g(Y,Z)X - (n-1)g(X,Z)Y - g(Y,Z)X + g(X,Z)Y],$$
(3.6,23)

Finally, we get

$$R(X,Y)Z = [g(Y,Z)X - g(X,Z)Y]$$
(3.6.24)

The above equation (3.6.24) implies that M is of constant curvature 1 and consequently it is locally isometric to the unit sphere S^n (1).

This completes the proof of the theorem.

Corollary 3.6.1. If an n−dimensional Lorentzian para-Kenmotsu manifold M satisfies the condition $\tilde{L}(\xi,U).\bar{L} = 0$ then it is locally isometric to the unit sphere S^n (1) where $\tilde{L}$ denotes the concircular curvature tensor on n−dimensional Lorentzian para-Kenmotsu manifold M.

Definition 3.6.2. An n−dimensional Lorentzian para-Kenmotsu manifold M is said to be conharmonic semi-symmetric if

$$R(X,Y)\bar{L} = 0 .$$
(3.6.25)

where R is the curvature tensor and X, Y are vector fields on M.

Corollary 3.6.2. An n−dimensional Lorentzian para-Kenmotsu manifold M is conharmonic semi-symmetric if and only if it is conharmonically flat Lorentzian para-Kenmotsu manifold.

Proof. From **Theorem (3.6.1)** we know that every conharmonically flat Lorentzian para-Kenmotsu manifold M is an Einstein manifold. Also, we know that every Einstein manifold is conharmonic semi-symmetric but in general, the converse is not true. Here, we prove that in a Lorentzian para-Kenmotsu manifold $R(X,Y).\bar{L} = 0$, which implies that the manifold M is conharmonically flat.

If $R(X,Y).\bar{L} = 0$, we have

$$R(X,Y)\bar{L})(U,V)W = R(X,Y)\bar{L}(U,V)W$$
(3.6.26)

$$-\bar{L}(R(X,Y)U,V)W - \bar{L}(U,R(X,Y)V)W - \bar{L}(U,V)R(X,Y)W = 0$$

Taking the inner product of the above equation (3.6.26) with ξ we obtain

$$0 = g(R(X,Y)\bar{L}(U,V)W,\xi)$$

$$-g(\bar{L}(R(X,Y)U,V)W,\xi) - g(\bar{L}(U,R(X,Y)V)W,\xi) - g(\bar{L}(U,V)R(X,Y)W,\xi),$$

Using (3.2.1), (3.2.5) and (3.2.8) we have

$$0 = -\bar{L}(U,V,W,Y - \eta(Y)\eta(\bar{L}(U,V)W) + \eta(U)\eta(\bar{L}(Y,V)W + \eta(V)\eta\bar{L}(U,Y)W) + \eta(W)\eta(\bar{L}(U,V)Y),$$

On simplification above equation becomes

$$\bar{L}(U,V,W,Y) = 0.$$

Therefore, M is conharmonically flat Lorentzian para-Kenmotsu manifold.

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