Mathematical Modeling of a Transverse Shear Deformation Thick Shell Theory

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Abstract— **Three-dimensional theory (3D) of elasticity in curvilinear coordinates is employed to understand the stress and strain distribution in the middle surface of a thick composite shell under various operating conditions. The equations of motion are derived by making use of the relationships between forces, moments and stress displacements of shell using Hamilton's principle of minimum energy. The necessary theoretical assumptions are discussed to simplify the three dimensions to a set of two-dimensional (2D) shell equations without violating the theory of elasticity. Displacement through the thickness is of third order in this analysis as compared with the first order approximation as previously published research. Equilibrium equations are formulated using these equations to achieve a set of linear partial differential equations and solved for exact solutions using Fourier series expansion for simply supported laminated cross-ply boundaries. This solution can be further used for various vibration analysis and optimal design of thick shell structures. Finally, the new additional parameters using the third order shear deformation of thick shell theory obtained from Fourier expansion is compared with the first order shear deformation shell theory from the literature.**

Index Terms—**Shells, Thick, Shear Deformation, Laminated, Exact Solution, Cross-ply**

1. INTRODUCTION

A shell is a three dimensional body bounded by two close surfaces. Its thickness is constant if these surfaces are parallel to each other. In general, we can assume that the thickness (i.e. the distance between these two surfaces) is much smaller in comparison to its length, width and radii of the curvature. Considerable amount of works been done on shell theory (Koiter, 1969; Love, 1892; Reissner, 1945; Ye & Soldatos, 1994; Qatu et al., 2012). There are some differences in the results obtained by various researchers depending upon the assumptions made, the resulting theory derived, and the size of the parameters. The solution procedure used in solving the equations of motion incorporates the stress–strain, the strain–displacement and the associated boundary conditions of the shell.

Based on the classical linear elasticity, Love was the first person to present a successful shell theory (Qatu, 1999; Love, 1892; Ventsel & Krauthammer, 2001). Love was able to unitize Kirchhoff hypotheses to simplify the relationship between strain displacement and constitutive equations, in the case of the plate bending theory having small deflection and thinness (Ventsel & Krauthammer, 2001). These are the well-known Kirchhoff–Love's assumptions (Koiter, 1969; Love, 1892). Love's assumptions of thin elasticity shells are referred to as the classical laminated shell theory.

Love's theory had certain shortcomings even its well accepted popularity and success in the applications of shell structures, due to the incorporation of infinitesimal strains of bending (small terms) and extension. However, in Love's theory, only few of these terms in the model equations were sustained. Thus, in Love's theory of differential operator matrix, the displacement of equilibrium equations of certain shells became asymmetric. However, later the inconsistencies of Love's theory (Timoshenko & Woinowsky, 1959) were modified by Reissner, (1945) mathematical model for two-dimensional linear thin shell theory using Love-Kirchhoff hypotheses. "Reissner derived the equations of equilibrium, strain–displacement relations, and stress resultants expressions for thin shells directly from the 3D theory of elasticity by applying the Love–Kirchhoff hypotheses and neglecting small terms of order

z/R_i ($i=1,2$) (Vlasov, 1964)".

Nomenclature

$A_{ij}, \hat{A}_{ij}, \overline{A}_{ij}$ *stretching and shearing stiffness parameters*

$A_{ij_\alpha}, A_{ij_\beta}, A_{ij_n}$ *stiffness parameters*

$B_{ij}, \hat{B}_{ij}, \overline{B}_{ij}$ *coupling stiffness parameters*

$B_{ij_\alpha}, B_{ij_\beta}, B_{ij_n}$ *stiffness parameters*

c_0, c_1 *tracer*

$D_{ij}, \hat{D}_{ij}, \overline{D}_{ij}$ *bending and twisting stiffness parameters*

$D_{ij_\alpha}, D_{ij_\beta}, D_{ij_n}$ *stiffness parameters*

$E_{ij}, F_{ij}, L_{ij}, E_{ij_\alpha}, E_{ij_\beta}, E_{ij_n}, F_{ij_\alpha}, F_{ij_\beta}, F_{ij_n}, L_{ij_\alpha}, L_{ij_\beta}, L_{ij_n}$ *higher order stiffness parameters*

t *time*

I_i *rotary inertia*

$\rho^{(k)}$ *mass density of kth layer*

$\sigma_\alpha, \sigma_\beta, \sigma_z$ *normal stress*

$\sigma_{\alpha\beta}, \sigma_{\beta z}, \sigma_{\alpha z}$ *shear stress*

$\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_z$ *normal strains*

$\gamma_{\alpha\beta}, \gamma_{\alpha z}, \gamma_{\beta z}$ *shear strains*

$\overline{Q}_{ij}^{(k)}, \overline{Q_{ij}}$ *elastic stiffness parameters for layer k*

Q_α, Q_β *transverse shear force*

$m_\alpha^{(1)}, m_\beta^{(1)}, m_\alpha^{(2)}, m_\beta^{(2)}$ *distributed couples*

q_z, q_β, q_α *distributed forces*

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In order to develop a better thick shell theory, we re-examined Love's first order approximation theory of thin shell. Various studies (Qatu, 2004; Noor, 1990; Leissa, 1973) concluded that even for thick shells, in comparison to other stresses and strains, the transverse normal strain and stress remains small enough to be neglected (Love, 1892).
Many shell theories and associated problems were derived based on Love's first approximation as mentioned above. However, during the middle of the 20th century (Leissa. 1973; Qatu, 2002), many of these theories found to be inconsistent, as reported in the literature on shell theory e.g. Timoshenko & Woinowsky (1959). The introduction of unsymmetrical differential operators by rigid body motion contradicts the theory of reciprocity thus in free vibration analysis, which yields natural frequencies in complex domain. Moreover, other inconsistencies were introduced due to the assumption of small thickness ($\frac{z}{R} << 1$ and h/R) and also due to symmetric stress resultants (i.e., $N_{\alpha\beta} = N_{\beta\alpha}$ and $M_{\alpha\beta} = M_{\beta\alpha}$) of the shell. In the case of non-spherical geometrical structures, the stress resultants are not always equal. In order to overcome such deficiencies, various theoretical treatments have been proposed by several researchers. However, Vlasov (1964) resolved these discrepancies by extending the strain displacement term. Usually the terms in stress resultant equation expansion using Taylor series have negligible terms in the denominator (Noor, 1990; Qatu, 1999, 2002). Various survey articles on these topics were implemented on the behavior of homogeneous and laminated composite shells (Ambartsumian, 1962; Gol'denveizer, 1961; Timoshenko & Woinowsky 1959; Love, 1892). Earlier theories and analyses about the effects of laminated shear deformation composite materials allowed its importance in shell theory when compared with isotropic materials (Qatu, 2004).Often, higher order shear shell deformation theory included higher order approximation, arising from stress-strain and rotational inertia. However, those who worked earlier, on shear deformation shell theories, failed to consider (1+z/R) term in the equations of stress resultant as discussed above. This was reported by Bert, (1967) and followed by Qatu et al. (2004), since it increased the inaccuracies in the stress-strain equations for the thick laminated shells. These results were obtained either by integrating the above equations exactly or by Taylor series expansion, which showed good agreement with the 3D elasticity theory (Qatu et al., 2012). Higher order terms of the shear strain, and rotary inertia approximation were used by many others in the shear deformation theories but again neglected (1+z/R) in the stress resultant equations (Kapania, 1989; Khare et al., 2005; Qatu et al., 2010), which is only applicable to shallow shells. These inaccuracies led to the development of constitutive equations and were observed by many researchers (Ye, JQ. & Soldatos, 1994; Kant & Swaminathan, 2001; Qatu, 1994; Qatu et al., 2010). However, Leissa (1973) used the geometric series to truncate these terms and Qatu et al (2010, 2012) integrated this term exactly and the resulting results were in good agreement with 3D theory of elasticity. Others who had worked on the same problem, although included higher order terms of shear strains, yet once again neglecting the contribution of the term (1+z/R) and mostly applied to shallow shells rather than deep shells. Therefore, the inclusion of this term in our present article would add value to the existing theories and can present a better approximation for thick shell under various operating conditions.
Due to the importance of the shells of revolution in the development of thin shell theory, contribution of (1+z/R) term cannot be waved aside. This has various applications in engineering. However, Reissner (1941, 1945) developed a spherical shell theory and obtained a classical derivation of the bending problem for shells of revolution. He used asymptotic method for integration after reducing the resulting differential equations of the spherical shell. Flügge (1962) worked on the spherical and conical shells and, was able to derive general solutions and the result was based on the classical displacement method.
In short, a complete and general thin elastic shell theory from the theory of linear elasticity was first developed almost one hundred years ago by Love (1892). Prior to this development, Aron (1874) presented a model equation for bending of thin shells using the Kirchhoff and Clebsch's small strain and finite displacement theory. However, the erroneous mathematical treatment and the justification given in Aron (1874) model led to Love's theory of thin shells. Since then, many new developments in the treatment of shell theory for varied shape, size, and method of solution and its interpretation aroused from various researchers in this field (Ventsel & Krauthammer, 2001; Reddy, 1984; Ye, & Soldatos, 1994; Qatu et al., 2013). Towards this end, in this paper, we propose our contribution towards the mathematical theory of third order shear deformation thick shell theory by Zannon (TSDTZ) and its stress-strain deformation at the mid thick shell surface.

2. Mathematical and Theoretical Assumptions

2.1. Displacement Models.

In the present work we consider a thick shell having smaller thickness in comparison with other shell parameters such as width, shape length and curvature radii, which is usually taken as (1/10) of its measure. In any vibrational analysis, thick shells often includes rotational inertia factors and shear deformation (Qatu, 2004, 2012). Middle plane displacements are stretched in terms of shell thickness in shell deformation theory and can make use of first or higher order approximation. Therefore, three dimensional elasticity theories are reduced to two dimensional theories by neglecting the normal strain in comparison with other components of the strain, which are acting on the plane parallel to the middle surface. Generally, such assumption can be justified outside of the neighborhood of highly rigorous force, hence the stretching in the z-direction can be ignored, which leads to zero strain in the azimuthal (z) direction. Hence the displacements are written in the following polynomial form in variable thickness (z)

$$\begin{aligned} u(\alpha,\beta,z) &= u_0(\alpha,\beta) + z\psi_\alpha(\alpha,\beta) + z^3\varphi_\alpha(\alpha,\beta) \\ v(\alpha,\beta,z) &= v_0(\alpha,\beta) + z\psi_\beta(\alpha,\beta) + z^3\varphi_\beta(\alpha,\beta) \\ w(\alpha,\beta,z) &= w_0(\alpha,\beta) + z\psi_z(\alpha,\beta). \end{aligned} \quad (2.1)$$

Where $\frac{-h}{2} \le z \le \frac{h}{2}, h$ is the shell thickness, u_0, v_0 and w_0 are middle surface displacements of the shell and $\psi_\alpha, \psi_\beta, \psi_z$ are middle surface rotations and $\varphi_\alpha, \varphi_\beta$ are higher order terms rotation of transverse normal. In the third part of the equations (2.1), we are assuming $\varepsilon_z \neq 0$. By substituting equation (2.1) into the strain-displacement equation of the elasticity theory, the following strain-displacement equations are obtained:

$$\varepsilon_\alpha = \frac{1}{(1+ z/R_\alpha)}(\varepsilon_{0\alpha} + z\kappa_\alpha^{(1)} + z^2\kappa_\alpha^{(2)}), \ \varepsilon_\beta = \frac{1}{(1+ z/R_\beta)}(\varepsilon_{0\beta} + z\kappa_\beta^{(1)} + z^2\kappa_\beta^{(2)})$$

$$\varepsilon_z = \psi_z(\alpha,\beta) \neq 0, \quad \varepsilon_{\alpha\beta} = \frac{1}{(1+ z/R_\alpha)}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}^{(1)} + z^2\kappa_{\alpha\beta}^{(2)}) \tag{2.2}$$

$$\varepsilon_{\beta\alpha} = \frac{1}{(1+ z/R_\beta)}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}^{(1)} + z^2\kappa_{\beta\alpha}^{(2)}), \ \gamma_{\alpha z} = \frac{1}{(1+ z/R_\alpha)}(\gamma_{0\alpha z} + zG^{(1)} + z^2G^{(2)})$$

$$\gamma_{\beta z} = \frac{1}{(1+ z/R_\beta)}(\gamma_{0\beta z} + zE^{(1)} + z^2E^{(2)}).$$

Where the middle surface strains are

$$\varepsilon_{0\alpha} = \frac{1}{A}\frac{\partial u_0}{\partial\alpha} + \frac{v_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_\alpha}, \ \varepsilon_{0\beta} = \frac{1}{B}\frac{\partial v_0}{\partial\beta} + \frac{u_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_\beta},$$

$$\varepsilon_{0\alpha\beta} = \frac{1}{A}\frac{\partial v_0}{\partial\alpha} - \frac{u_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_{\alpha\beta}}, \ \varepsilon_{0\beta\alpha} = \frac{1}{B}\frac{\partial u_0}{\partial\beta} - \frac{v_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_{\alpha\beta}}, \tag{2.3}$$

$$\gamma_{0\alpha z} = \frac{1}{A}\frac{\partial w_0}{\partial\alpha} - \frac{u_0}{R_\alpha} - \frac{v_0}{R_{\alpha\beta}} + \psi_\alpha, \ \gamma_{0\beta z} = \frac{1}{B}\frac{\partial w_0}{\partial\beta} - \frac{v_0}{R_\beta} - \frac{u_0}{R_{\alpha\beta}} + \psi_\beta.$$

The curvature and twist changes of the shells can be written as

$$\kappa_\alpha^{(1)} = \frac{1}{A}\frac{\partial\psi_\alpha}{\partial\alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial\beta} + \frac{\psi_z}{R_\alpha}, \ \kappa_\beta^{(1)} = \frac{1}{A}\frac{\partial\psi_\beta}{\partial\beta} + \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial\alpha} + \frac{\psi_z}{R_\beta},$$

$$\kappa_\alpha^{(2)} = \frac{1}{A}\frac{\partial\phi_\alpha}{\partial\alpha} + \frac{\phi_\beta}{AB}\frac{\partial A}{\partial\beta}, \ \kappa_\beta^{(2)} = \frac{1}{A}\frac{\partial\phi_\beta}{\partial\beta} + \frac{\phi_\alpha}{AB}\frac{\partial B}{\partial\alpha}, \tag{2.4}$$

$$\kappa_{\alpha\beta}^{(1)} = \frac{1}{A}\frac{\partial\psi_\beta}{\partial\alpha} - \frac{\psi_\alpha}{AB}\frac{\partial A}{\partial\beta} + \frac{\psi_z}{R_{\alpha\beta}}, \ \kappa_{\alpha\beta}^{(2)} = \frac{1}{A}\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\phi_\alpha}{AB}\frac{\partial A}{\partial\beta}.$$

$$\kappa_{\beta\alpha}^{(1)} = \frac{1}{B}\frac{\partial\psi_\alpha}{\partial\beta} - \frac{\psi_\beta}{AB}\frac{\partial B}{\partial\alpha} + \frac{\psi_z}{R_{\alpha\beta}}, \kappa_{\beta\alpha}^{(2)} = \frac{1}{B}\frac{\partial\phi_\alpha}{\partial\beta} - \frac{\phi_\beta}{AB}\frac{\partial B}{\partial\alpha}.$$

Here,

$$G^{(1)} = \frac{1}{A}\frac{\partial\psi_z}{\partial\alpha} + 2\phi_\alpha - \frac{\psi_\beta}{R_{\alpha\beta}}, \quad G^{(2)} = \frac{\phi_\beta}{R_\alpha} - \frac{\phi_\alpha}{R_{\alpha\beta}}.$$

$$E^{(1)} = \frac{1}{B}\frac{\partial\psi_z}{\partial\beta} + 2\phi_\beta - \frac{\psi_\alpha}{R_{\alpha\beta}}, \quad E^{(2)} = \frac{\phi_\beta}{R_\beta} - \frac{\phi_\alpha}{R_{\alpha\beta}}.$$

The contribution of the terms $zG^{(1)}, z^2G^{(2)}, zE^{(1)}$ and $z^2E^{(2)}$ to the overall accuracy of the theory is presented in the later section. Equations (2.2) - (2.4) establish the stress-strain displacement relationship required for 3rd order shear deformation shell theory, which are different from previous researchers (Qatu et al., 2010, 2012; Ye & Soldatos, 1994; Reddy, 1984; Ventsel & Krauthammer, 2001).

After integrating the stress over the thickness of shell by incorporating the term $(1+\frac{z}{R})$, we get the following moment and force resultant equations.

$$\begin{Bmatrix} N_\alpha \\ N_{\alpha\beta} \\ Q_\alpha \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + z/R_\beta\right) dz, \quad \begin{Bmatrix} N_\beta \\ N_{\beta\alpha} \\ Q_\beta \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + z/R_\alpha\right) dz. \tag{2.5}$$

Similarly, the bending and twisting moment resultants are defined as

$$\begin{Bmatrix} M_\alpha^{(1)} \\ M_{\alpha\beta}^{(1)} \\ P_\alpha^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + {z}/{R_\beta}\right) z\, dz\,, \quad \begin{Bmatrix} M_\beta^{(1)} \\ M_{\beta\alpha}^{(1)} \\ P_\beta^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + {z}/{R_\alpha}\right) z\, dz$$

$$\begin{Bmatrix} M_\alpha^{(2)} \\ M_{\alpha\beta}^{(2)} \\ P_\alpha^{(2)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + {z}/{R_\beta}\right) z^2\, dz, \quad \begin{Bmatrix} M_\beta^{(2)} \\ M_{\beta\alpha}^{(2)} \\ P_\beta^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + {z}/{R_\alpha}\right) z^2\, dz\,. \tag{2.6}$$

Where $P_\alpha^{(1)}, P_\alpha^{(2)}, P_\beta^{(1)}$ and $P_\beta^{(2)}$ are higher order shear resultants terms and

$$\sigma_\alpha = \overline{Q_{11}}\,\varepsilon_\alpha + \overline{Q_{12}}\,\varepsilon_\beta + \overline{Q_{13}}\,\varepsilon_z + \overline{Q_{16}}\,\gamma_{\alpha\beta}\,.$$
$$\sigma_\beta = \overline{Q_{12}}\,\varepsilon_\alpha + \overline{Q_{22}}\,\varepsilon_\beta + \overline{Q_{23}}\,\varepsilon_z + \overline{Q_{26}}\,\gamma_{\alpha\beta}\,.$$
$$\sigma_z = \overline{Q_{13}}\,\varepsilon_\alpha + \overline{Q_{23}}\,\varepsilon_\beta + \overline{Q_{33}}\,\varepsilon_z + \overline{Q_{36}}\,\gamma_{\alpha\beta}\,.$$

$$\sigma_{\beta z} = \overline{Q_{44}}\,\gamma_{\beta z} + \overline{Q_{45}}\,\gamma_{\alpha z}\,.$$
$$\sigma_{\alpha z} = \overline{Q_{45}}\,\gamma_{\beta z} + \overline{Q_{55}}\,\gamma_{\alpha z}\,.$$
$$\sigma_{\alpha\beta} = \overline{Q_{16}}\,\varepsilon_\alpha + \overline{Q_{26}}\,\varepsilon_\beta + \overline{Q_{36}}\,\varepsilon_z + \overline{Q_{66}}\,\gamma_{\alpha\beta}\,.$$

It is noteworthy that, though the stresses $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ are the same, but the stress resultants $N_{\alpha\beta}$ and $N_{\beta\alpha}$ are not (Qatu, 2004). These stress resultants are only equal $(N_{\alpha\beta} = N_{\alpha\beta};\ M_{\alpha\beta}^{(1)} = M_{\beta\alpha}^{(1)};\ M_{\alpha\beta}^{(2)} = M_{\beta\alpha}^{(2)})$ if the radius of curvature, $R_\alpha = R_\beta$, which is the conditions of spherical shells.

Therefore, the stress resultants obtained from the above equations (2.5)-(2.6) can be rewritten in the following matrix form:

$$\begin{bmatrix} N_\alpha \\ N_\beta \\ N_z \\ N_{\alpha\beta} \\ N_{\beta\alpha} \\ M_\alpha^{(1)} \\ M_\beta^{(1)} \\ M_{\alpha\beta}^{(1)} \\ M_{\beta\alpha}^{(1)} \\ M_\alpha^{(2)} \\ M_\beta^{(2)} \\ M_{\alpha\beta}^{(2)} \\ M_{\beta\alpha}^{(2)} \end{bmatrix} = \begin{bmatrix}
\bar{A}_{11} & A_{12} & A_{13} & \bar{A}_{16} & A_{16} & \overline{B_{11}} & B_{12} & \overline{B_{16}} & B_{16} & \overline{D_{11}} & D_{12} & \overline{D_{16}} & D_{16} \\
A_{12} & \hat{A}_{22} & A_{23} & A_{26} & \hat{A}_{26} & B_{12} & \hat{B}_{22} & B_{26} & \hat{B}_{26} & D_{12} & \hat{D}_{22} & D_{26} & \hat{D}_{26} \\
A_{13} & A_{23} & A_{33} & \overline{A_{36}} & A_{36} & B_{13} & B_{23} & \overline{B_{36}} & B_{36} & D_{13} & D_{23} & \overline{D_{36}} & D_{36} \\
\bar{A}_{16} & A_{26} & \bar{A}_{36} & \bar{A}_{66} & A_{66} & \bar{B}_{16} & B_{26} & \bar{B}_{66} & B_{66} & \bar{D}_{16} & D_{26} & \bar{D}_{66} & D_{66} \\
A_{16} & \hat{A}_{26} & A_{36} & A_{66} & \hat{A}_{66} & B_{16} & \hat{B}_{26} & B_{66} & \hat{B}_{66} & D_{16} & \hat{D}_{26} & D_{66} & \hat{D}_{66} \\
\bar{B}_{11} & B_{12} & B_{13} & \bar{B}_{16} & B_{16} & \bar{D}_{11} & D_{12} & \bar{D}_{16} & D_{16} & \bar{E}_{11} & E_{12} & \bar{E}_{16} & E_{16} \\
B_{12} & \hat{B}_{22} & B_{23} & B_{26} & \hat{B}_{26} & D_{12} & \hat{D}_{22} & D_{26} & \hat{D}_{26} & E_{12} & \hat{E}_{22} & E_{26} & \hat{E}_{26} \\
\bar{B}_{16} & B_{26} & \bar{B}_{36} & \bar{B}_{66} & B_{66} & \bar{D}_{16} & D_{26} & \bar{D}_{66} & D_{66} & \bar{E}_{16} & E_{26} & \bar{E}_{66} & E_{66} \\
B_{16} & \hat{B}_{26} & B_{36} & B_{66} & \hat{B}_{66} & D_{16} & \hat{D}_{26} & D_{66} & \hat{D}_{66} & E_{16} & \hat{E}_{26} & E_{66} & \hat{E}_{66} \\
\bar{D}_{11} & D_{12} & D_{13} & \bar{D}_{16} & D_{16} & \bar{E}_{11} & E_{12} & \bar{E}_{16} & E_{16} & \bar{F}_{11} & F_{12} & \bar{F}_{16} & F_{16} \\
D_{12} & \hat{D}_{22} & D_{23} & D_{26} & \hat{D}_{26} & E_{12} & \hat{E}_{22} & E_{26} & \hat{E}_{26} & F_{12} & \hat{F}_{22} & F_{26} & \hat{F}_{26} \\
\bar{D}_{16} & D_{26} & \bar{D}_{36} & \bar{D}_{66} & D_{66} & \bar{E}_{16} & E_{26} & \bar{E}_{66} & E_{66} & \bar{F}_{16} & F_{26} & \bar{F}_{66} & F_{66} \\
D_{16} & \hat{D}_{26} & D_{36} & D_{66} & \hat{D}_{66} & E_{16} & \hat{E}_{26} & E_{66} & \hat{E}_{66} & F_{16} & \hat{F}_{26} & F_{66} & \hat{F}_{66}
\end{bmatrix} \begin{bmatrix} \varepsilon_{0\alpha} \\ \varepsilon_{0\beta} \\ \varepsilon_{0z} \\ \varepsilon_{0\alpha\beta} \\ \varepsilon_{0\beta\alpha} \\ \kappa_\alpha^{(1)} \\ \kappa_\beta^{(1)} \\ \kappa_{\alpha\beta}^{(1)} \\ \kappa_{\beta\alpha}^{(1)} \\ \kappa_\alpha^{(2)} \\ \kappa_\beta^{(2)} \\ \kappa_{\alpha\beta}^{(2)} \\ \kappa_{\beta\alpha}^{(2)} \end{bmatrix} \tag{2.7}$$

$$\begin{bmatrix} Q_\alpha \\ Q_\beta \\ P_\alpha^{(1)} \\ P_\beta^{(1)} \\ P_\alpha^{(2)} \\ P_\beta^{(2)} \end{bmatrix} = \begin{bmatrix}
\bar{A}_{55} & A_{45} & \bar{B}_{55} & B_{45} & \bar{D}_{55} & D_{45} \\
A_{45} & \hat{A}_{44} & B_{45} & \hat{B}_{44} & D_{45} & \hat{D}_{44} \\
\bar{B}_{55} & B_{45} & \bar{D}_{55} & D_{45} & \bar{E}_{55} & E_{45} \\
B_{45} & \hat{B}_{44} & D_{45} & \hat{D}_{44} & E_{45} & \hat{E}_{44} \\
\bar{D}_{55} & D_{45} & \bar{E}_{55} & E_{45} & \bar{F}_{55} & F_{45} \\
D_{45} & \hat{D}_{44} & E_{45} & \hat{E}_{44} & F_{45} & \hat{F}_{44}
\end{bmatrix} \begin{bmatrix} \gamma_{0\alpha z} \\ \gamma_{0\beta z} \\ G^{(1)} \\ E^{(1)} \\ G^{(2)} \\ E^{(2)} \end{bmatrix}. \tag{2.8}$$

Where $A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, \bar{A}_{ij}, \bar{B}_{ij}, \bar{D}_{ij}, \bar{E}_{ij}, \bar{F}_{ij}, \hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}, \hat{E}_{ij}$ and $\hat{F}_{ij}$ are defined below.

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h_k - h_{k-1}),\ B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h^2{}_k - h^2{}_{k-1}),$$
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h^3{}_k - h^3{}_{k-1}),\ E_{ij} = \frac{1}{4}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h^4{}_k - h^4{}_{k-1}), \tag{2.9}$$
$$F_{ij} = \frac{1}{5}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (h^5{}_k - h^5{}_{k-1})$$

$$\bar{A}_{ij} = A_{ij\alpha} + \frac{B_{ij\alpha}}{R_\beta},\ \hat{A}_{ij} = A_{ij\beta} + \frac{B_{ij\beta}}{R_\alpha},\ \bar{B}_{ij} = B_{ij\alpha} + \frac{D_{ij\alpha}}{R_\beta},$$
$$\hat{B}_{ij} = B_{ij\beta} + \frac{D_{ij\beta}}{R_\alpha},\ \bar{D}_{ij} = D_{ij\alpha} + \frac{E_{ij\alpha}}{R_\beta}, \hat{D}_{ij} = D_{ij\beta} + \frac{E_{ij\beta}}{R_\alpha}, \tag{2.10}$$
$$\bar{E}_{ij} = E_{ij\alpha} + \frac{F_{ij\alpha}}{R_\beta},\ \hat{E}_{ij} = E_{ij\beta} + \frac{F_{ij\beta}}{R_\alpha},\ \bar{F}_{ij} = F_{ij\alpha} + \frac{L_{ij\alpha}}{R_\beta},\ \hat{F}_{ij} = F_{ij\beta} + \frac{L_{ij\beta}}{R_\alpha}.$$

Where $A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}$ are defined as follows:

$$A_{ij} = \sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h_k - h_{k-1}), B_{ij} = \frac{1}{2}\sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h^2{}_k - h^2{}_{k-1})$$
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h^3{}_k - h^3{}_{k-1}),\ E_{ij} = \frac{1}{4}\sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h^4{}_k - h^4{}_{k-1}) \tag{2.11}$$
$$F_{ij} = \frac{1}{5}\sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h^5{}_k - h^5{}_{k-1}). \qquad \forall\ i, j = 4, 5.$$

Where K_i and K_j are shear correction coefficients in i and j directions, $h_k - h_{k-1}$ is the thickness of the k^{th} layer of the composite shell and

$$A_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{\bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz,\ B_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{z \cdot \bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz,$$
$$D_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{z^2 \cdot \bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz,\ E_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{z^3 \cdot \bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz, \tag{2.12}$$
$$F_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{z^4 \cdot \bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz,\ L_{ijn} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \frac{z^5 \cdot \bar{Q}_{ij}^{(k)}}{1 + z/R_n} dz. \quad \forall\ n = \alpha, \beta$$

In order to obtain better numerical stability, we truncated the equations described in (2.12) and the term $1/(1+(z/R_n))$ in equation (2.12), which can be written in geometric series as

$$\frac{1}{1 + z/R_n} = 1 - \frac{z}{R_n} + \frac{z^2}{R_n{}^2} - \frac{z^3}{R_n{}^3} + (\text{Higher order terms}). \tag{2.13}$$

The term higher than $O(\frac{z}{R_n})^3$ are neglected to reduce the mathematical complexity, which is used in equations (2.9)-(2.12) and obtained the following form and is similar to the derivation given in the references (Qatu et al., 1999, 2010, 2012) for comparison.

$$\begin{aligned}
&\overline{A}_{ij} = A_{ij} - c_0 B_{ij} + c_1 D_{ij},\ \hat{A}_{ij} = A_{ij} + c_0 B_{ij} - c_1 D_{ij},\\
&\overline{B}_{ij} = B_{ij} - c_0 D_{ij} + c_1 E_{ij},\ \hat{B}_{ij} = B_{ij} + c_0 D_{ij} - c_1 E_{ij},\\
&\overline{D}_{ij} = D_{ij} - c_0 E_{ij} + c_1 F_{ij},\ \hat{D}_{ij} = D_{ij} + c_0 E_{ij} - c_1 F_{ij},\\
&\overline{E}_{ij} = E_{ij} - c_0 F_{ij} + c_1 L_{ij},\ \hat{E}_{ij} = E_{ij} + c_0 F_{ij} - c_1 L_{ij}. \qquad \forall i = 1,2,3,4,5,6.
\end{aligned} \tag{2.14}$$

And where

$$c_0 = \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right) \ \ and \ \ c_1 = \left(\frac{1}{R_\alpha^2} - \frac{1}{R_\alpha R_\beta}\right).$$

2.2. Hamilton's Principle: Equations of Motion and Boundary Conditions.

The behavior of elastic vibrations, excitation and its inherent properties on arbitrary bounded shells are often difficult to solve from the elastodynamic theory. Making various dynamics assumptions about the motion of the shell surfaces under numerous conditions can be used to solve these problems, one such method is the vibrational or Hamiltonian principles of minimum energy. Hamiltonian in some approximate form can be used to construct surface vibrations. With the help of Hamilton's principle various physical mechanisms such as perturbation in motion, rotary inertia, twist and shear distortion can be analyzed (Bert & Baker, 1969; Ames & van der Houwen, 1992; Love, 1892).

The Hamilton's principle (Ye & Soldatos, 1994Khare et al., 2005) of minimum energy of the governing equation is

$$\delta \int_{t_0}^{t_1} (K + W - U)dt = 0. \tag{2.15}$$

Where U is the strain energy, K the kinetic energy and W the external work by the system. The total strain energy and the middle surface stress and strains resultants of the shell have the following relationship:

$$\begin{aligned}
U &= \frac{1}{2}\int_V \{\sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \sigma_z \varepsilon_z + \sigma_{\alpha\beta}\varepsilon_{\alpha\beta} + \sigma_{\alpha z}\gamma_{\alpha z} + \sigma_{\beta z}\gamma_{\beta z}\}dV\\
&= \frac{1}{2}\int_\alpha\int_\beta \{N_\alpha \varepsilon_{0\alpha} + N_\beta \varepsilon_{0\beta} + N_z \varepsilon_{0z} + N_{\alpha\beta}\varepsilon_{0\alpha\beta} + N_{\beta\alpha}\varepsilon_{0\beta\alpha} + M_\alpha^{(1)}\kappa_\alpha^{(1)} + M_\alpha^{(2)}\kappa_\alpha^{(2)}\\
&+ M_\beta^{(1)}\kappa_\beta^{(1)} + M_\beta^{(2)}\kappa_\beta^{(2)} + M_{\alpha\beta}^{(1)}\kappa_{\alpha\beta}^{(1)} + M_{\alpha\beta}^{(2)}\kappa_{\alpha\beta}^{(2)} + M_{\beta\alpha}^{(1)}\kappa_{\beta\alpha}^{(1)} + M_{\beta\alpha}^{(2)}\kappa_{\beta\alpha}^{(2)} + Q_\alpha \gamma_{0\alpha z}\\
&+ Q_\beta \gamma_{0\beta z} + P_\alpha^{(1)}G^{(1)} + P_\alpha^{(2)}G^{(2)} + P_\beta^{(1)}E^{(1)} + P_\beta^{(2)}E^{(2)}\}AB d\alpha\, d\beta.
\end{aligned} \tag{2.16}$$

The total external work and the total kinetic energy of the thick shell

$$\begin{aligned}
W = \int_\alpha\int_\beta \{&q_\alpha u_0 + q_\beta v_0 + q_n w_0 + m_\alpha^{(1)}G^{(1)} + m_\alpha^{(2)}G^{(2)} + m_z\psi_z\\
&+ m_\beta^{(1)}E^{(1)} + m_\beta^{(2)}E^{(2)}\}AB\, d\alpha\, d\beta.
\end{aligned}$$

$$\begin{aligned}
T &= \frac{1}{2}\int_V \{u^2 + v^2 + w^2\}dV\\
&= \frac{1}{2}\int_\alpha\int_\beta \ \{(u_0{}^2 + v_0{}^2 + w_0{}^2)\left(I_1 + I_2(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_3}{R_\alpha R_\beta}\right)\\
&+ (\psi_\alpha^2 + \psi_z^2)\left(I_3 + I_4(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_5}{R_\alpha R_\beta}\right)\\
&+ 2(u_0\psi_\alpha + v_0\psi_\beta + w_0\psi_z)\left(I_2 + I_3(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_4}{R_\alpha R_\beta}\right)\\
&+ 2(u_0\varphi_\alpha + v_0\varphi_\beta + \psi_\beta^2)\left(I_4 + I_5(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_6}{R_\alpha R_\beta}\right)\\
&+ 2(\varphi_\alpha\psi_\alpha + \varphi_\beta\psi_\beta)\left(I_5 + I_6(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_7}{R_\alpha R_\beta}\right)\\
&+ (\varphi_\alpha^2 + \varphi_\beta^2)\left(I_7 + I_8(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_9}{R_\alpha R_\beta}\right)\}AB\, dz\, d\alpha\, d\beta
\end{aligned} \tag{2.17}$$

Where the inertia terms are

$$\left[I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8,I_9\right]=\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\rho^{(k)}\left[1,z,z^2,z^3,z^4,z^5,z^6,z^7,z^8\right]dz \qquad (2.18)$$

In the above Hamilton's principle derivation, equations of motion and boundary conditions are derived by substituting the equations (2.16) and (2.17) in equation (2.15).

The resulting equations of motion are (Reddy, 1984; Qatu et al., 2013)

$$\frac{\partial}{\partial\alpha}(BN_\alpha)-\frac{\partial B}{\partial\alpha}N_\beta+\frac{\partial A}{\partial\beta}N_{\alpha\beta}+\frac{\partial}{\partial\beta}(AN_{\beta\alpha})+\frac{AB}{R_\alpha}Q_\alpha+\frac{AB}{R_{\alpha\beta}}Q_\beta+ABq_\alpha = AB(\overline{I}_1u_0+\overline{I}_2\psi_\alpha).$$

$$\frac{\partial}{\partial\alpha}(BN_{\alpha\beta})+\frac{\partial B}{\partial\alpha}N_{\beta\alpha}-\frac{\partial A}{\partial\beta}N_\alpha+\frac{\partial}{\partial\beta}(AN_\beta)+\frac{AB}{R_{\alpha\beta}}Q_\alpha+\frac{AB}{R_\beta}Q_\beta+ABq_\beta = AB(\overline{I}_1v_0+\overline{I}_2\psi_\beta).$$

$$\frac{\partial}{\partial\alpha}(BQ_\alpha)+\frac{\partial}{\partial\beta}(AQ_\beta)-AB\left(\frac{N_\alpha}{R_\alpha}+\frac{N_\beta}{R_\beta}+\frac{N_{\alpha\beta}+N_{\beta\alpha}}{R_{\alpha\beta}}\right)+ABq_n = AB(\overline{I}_1w_0).$$

$$\frac{\partial}{\partial\alpha}(BM_\alpha^{(1)})-\frac{\partial B}{\partial\alpha}M_\beta^{(1)}+\frac{\partial A}{\partial\beta}M_{\alpha\beta}^{(1)}+\frac{\partial}{\partial\beta}(AM_{\beta\alpha}^{(1)})-ABQ_\alpha+\frac{AB}{R_{\alpha\beta}}P_\beta^{(1)}+ABm_\alpha^{(1)} = AB(\overline{I}_2u_0+\overline{I}_3\psi_\alpha).$$

$$\frac{\partial}{\partial\beta}(AM_\beta^{(1)})-\frac{\partial A}{\partial\beta}M_\alpha^{(1)}+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta}^{(1)})+\frac{\partial B}{\partial\alpha}M_{\beta\alpha}^{(1)}-ABQ_\beta+\frac{AB}{R_{\alpha\beta}}P_\alpha^{(1)}+ABm_\beta^{(1)} = AB(\overline{I}_2v_0+\overline{I}_3\psi_\beta).$$

$$\frac{\partial}{\partial\alpha}(BP_\alpha^{(1)})+\frac{\partial}{\partial\beta}(AP_\beta^{(1)})-AB\left(N_z+\frac{M_\alpha^{(1)}}{R_\alpha}+\frac{M_\beta^{(1)}}{R_\beta}+\frac{M_{\alpha\beta}^{(1)}}{R_{\alpha\beta}}+\frac{M_{\beta\alpha}^{(1)}}{R_{\alpha\beta}}\right)+ABm_z = AB(\overline{I}_3\psi_z).$$

$$\frac{\partial}{\partial\alpha}(BM_\alpha^{(2)})-\frac{\partial B}{\partial\alpha}M_\beta^{(2)}+\frac{\partial}{\partial\beta}(AM_{\beta\alpha}^{(2)})-\left(2ABP_\alpha^{(1)}+\frac{AB}{R_{\alpha\beta}}P_\alpha^{(2)}+\frac{ABP_\beta^{(2)}}{R_{\alpha\beta}}\right)+ABm_\alpha^{(2)} = AB(\overline{I}_3u_0+\overline{I}_4\varphi_\alpha). \qquad (2.19)$$

$$\frac{\partial}{\partial\beta}(AM_\beta^{(2)})-\frac{\partial A}{\partial\beta}M_\alpha^{(2)}+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta}^{(2)})-\left(\frac{AB}{R_\beta}P_\beta^{(2)}+\frac{AB}{R_\alpha}P_\alpha^{(2)}+2ABP_\beta^{(1)}\right)+ABm_\beta^{(2)} = AB(\overline{I}_3v_0+\overline{I}_4\varphi_\beta).$$

Where $\overline{I}_i=\left(I_i+I_{i+1}\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)+\frac{I_{i+2}}{R_\alpha R_\beta}\right), \forall\ i=1,2,3,4$. (2.20)

The corresponding boundary conditions are given below for α, a constant

$$\begin{array}{llll}
either & N_{0\alpha}-N_\alpha=0 & or & u_0=0\\
either & N_{0\alpha\beta}-N_{\alpha\beta}=0 & or & v_0=0\\
either & Q_{0\alpha}-Q_\alpha=0 & or & w_0=0\\
either & M_{0\alpha}^{(1)}-M_\alpha^{(1)}=0 & or & \psi_\alpha=0\\
either & M_{0\alpha\beta}^{(1)}-M_{\alpha\beta}^{(1)}=0 & or & \psi_\beta=0\\
either & P_{0\alpha}^{(1)}-P_\alpha^{(1)}=0 & or & \psi_z=0\\
either & M_{0\alpha}^{(2)}-M_\alpha^{(2)}=0 & or & \varphi_\alpha=0\\
either & M_{0\alpha\beta}^{(2)}-M_{\alpha\beta}^{(2)}=0 & or & \varphi_\beta=0
\end{array} \qquad (2.21)$$

Similar boundary conditions are obtained by taking β, a constant.

3. Mathematical Analysis

It is known that there is no exact solution for a general lamination structure (shell, plate) with boundary conditions and/or lamination having series of sequence and layers. The exact solution of the equations of motion and boundary conditions for simply supported cross-ply thick shell (Flügge, 1962; Leissa, 1973; Ye & Soldatos, 1994) is formulated in the section. These partial differential equations of the motions (2.19) have solutions of the form (Librescu et al., 1989; Reddy, 1984; Reissner, 1945; Qatu et al., 2012, 2013)

$$u_0 = \sum_{m.n=1}^{N} u_{0,mn} Cos(A^*\alpha)\, Sin(B^*\beta)\, e^{-i\omega t}, \quad v_0 = \sum_{m.n=1}^{N} v_{0,mn} Sin(A^*\alpha)\, Cos(B^*\beta)\, e^{-i\omega t}$$

$$w_0 = \sum_{m.n=1}^{N} w_{0,mn} Sin(A^*\alpha)\, Sin(B^*\beta)\, e^{-i\omega t}, \quad \psi_\alpha = \sum_{m.n=1}^{N} \psi_{\alpha,mn} Cos(A^*\alpha)\, Sin(B^*\beta)\, e^{-i\omega t}$$

$$\psi_\beta = \sum_{m.n=1}^{N} \psi_{\beta,mn} Sin(A^*\alpha)\, Cos(B^*\beta)\, e^{-i\omega t}, \psi_z = \sum_{m.n=1}^{N} \psi_{z,mn} Sin(A^*\alpha)\, Sin(B^*\beta)\, e^{-i\omega t} \quad (3.1)$$

$$\phi_\alpha = \sum_{m.n=1}^{N} \phi_{\alpha,mn} Cos(A^*\alpha)\, Sin(B^*\beta)\, e^{-i\omega t}, \quad \phi_\beta = \sum_{m.n=1}^{N} \phi_{\beta,mn} Sin(A^*\alpha)\, Cos(B^*\beta)\, e^{-i\omega t}.$$

Where $A^* = \dfrac{m\pi}{a}, B^* = \dfrac{n\pi}{b}$ in which a and b are the dimensions of the middle part of the shell along the α - and β -axes, respectively.

Substituting these equations (3.1) into the differential equations of motion (2.19) yields a set of eight linear algebraic systems in terms of its respective components and collecting the coefficients, which can be written as an eigenvalue problem. Therefore, the resulting equations can be written in the following matrix form:

$$\{[L] - \lambda\, [M]\}\{\Delta\} = \{f\}. \quad (3.2)$$

Where $\lambda = \omega^2$, ω is the natural frequency and $\{\Delta\}$ is the displacement vector. The stiffness parameters L_{ij} and the mass parameter M_{ij} of the thick shell can be defined

$$L_{11} = -\overline{A}_{11} \cdot A^{*2} - \hat{A}_{66} \cdot B^{*2} + \frac{-\overline{A}_{55}}{(R_\alpha)^2} + \frac{-\hat{A}_{44}}{(R_{\alpha\beta})^2}, \; L_{12} = -A_{12} \cdot A^* \cdot B^* - A_{66} \cdot A^* \cdot B^*$$

$$L_{13} = \frac{\overline{A}_{11} \cdot A^*}{R_\alpha} + \frac{A_{12} \cdot A^*}{R_\beta} + \frac{\overline{A}_{55} \cdot A^*}{R_\alpha}, \; L_{14} = -\overline{B}_{11} \cdot A^{*2} - \hat{B}_{66} \cdot B^{*2} - \frac{\overline{A}_{55}}{R_\alpha} - \frac{\hat{B}_{44}}{(R_{\alpha\beta})}$$

$$L_{15} = -B_{12} \cdot A^* \cdot B^* - B_{66} \cdot A^* \cdot B^*, \; L_{16} = A_{13} \cdot A^* + \frac{\overline{B}_{11} \cdot A^*}{R_\alpha} + \frac{B_{12} \cdot A^*}{R_\beta} + \frac{\overline{B}_{55} \cdot A^*}{R_\alpha}$$

$$L_{17} = -\overline{D}_{11} \cdot A^{*2} - \hat{D}_{66} \cdot B^{*2} + \frac{2\,\overline{B}_{55}}{R_\alpha} - \frac{\overline{D}_{55}}{R_\alpha \cdot R_{\alpha\beta}} - \frac{\hat{D}_{44}}{(R_{\alpha\beta})^2}, \; L_{18} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*$$

$$L_{21} = -A_{12} \cdot A^* \cdot B^* - A_{66} \cdot A^* \cdot B^*, \; L_{22} = -\frac{\hat{A}_{22} \cdot B^{*2}}{R_\beta} - \overline{A}_{66} \cdot A^{*2} + \frac{\overline{A}_{55}}{(R_{\alpha\beta})^2} + \frac{\hat{A}_{44}}{(R_\beta)^2}$$

$$L_{23} = \frac{A_{12} \cdot B^*}{R_\alpha} + \frac{\hat{A}_{44} \cdot B^*}{R_\beta} + \hat{A}_{22} \cdot B^*, \; L_{24} = -B_{12} \cdot A^* \cdot B^* - B_{66} \cdot A^* \cdot B^*$$

$$L_{25} = -\hat{B}_{22} \cdot B^{*2} - \overline{B}_{66} \cdot A^{*2} - \frac{\overline{B}_{55}}{(R_{\alpha\beta})^2} + \frac{\hat{A}_{44}}{R_\beta}, \; L_{26} = A_{23} \cdot B^* + \frac{B_{12} \cdot B^*}{R_\alpha} + \frac{\hat{B}_{22} \cdot B^*}{R_\beta} + \frac{\hat{B}_{44} \cdot B^*}{R_\beta}$$

$$L_{27} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*, \; L_{28} = -\hat{D}_{22} \cdot B^{*2} - \overline{D}_{66} \cdot A^{*2} + \frac{\overline{D}_{55}}{R_\alpha \cdot R_{\alpha\beta}} + \frac{2 \cdot \hat{B}_{44}}{R_\beta} + \frac{\hat{D}_{44}}{(R_\beta)^2}$$

$$L_{31} = \frac{\overline{A}_{55} \cdot A^*}{R_\alpha} + \frac{\overline{A}_{11} \cdot A^*}{R_\alpha} + \frac{A_{12} \cdot A^*}{R_\beta}, \; L_{32} = \frac{\hat{A}_{44} \cdot B^*}{R_\beta} + \frac{A_{12} \cdot B^*}{R_\alpha} \quad (3.3)$$

$$L_{33} = -\overline{A}_{55} \cdot A^{*2} - \hat{A}_{44} \cdot B^{*2} - \frac{\overline{A}_{11}}{(R_\alpha)^2} - \frac{A_{12}}{(R_\alpha)^2} - \frac{A_{12}}{R_\alpha \cdot R_\beta} - \frac{(\overline{A}_{66} + 2 \cdot A_{66} + \hat{A}_{66})}{(R_{\alpha\beta})^2}$$

$$L_{34} = -\overline{A}_{55} \cdot A^* + \frac{\overline{B}_{11} \cdot A^*}{R_\alpha} + \frac{B_{12} \cdot A^*}{R_\beta}, \; L_{35} = -\hat{A}_{44} \cdot B^* + \frac{B_{12} \cdot B^*}{R_\alpha} + \frac{\hat{B}_{22} \cdot B^*}{R_\beta}$$

$$L_{36} = -A^{*2} \cdot \overline{B}_{55} - B^{*2}\,\hat{B}_{44} - \frac{A_{13}}{R_\alpha} - \frac{\overline{B}_{11}}{(R_\alpha)^2} - \frac{\hat{A}_{22}}{R_\beta} - \frac{2 \cdot B_{12}}{R_\alpha \cdot R_\beta} - \frac{(\overline{A}_{36} + A_{36})}{R_{\alpha\beta}} - \frac{(2 \cdot \overline{B}_{66} + 2 \cdot B_{66})}{(R_{\alpha\beta})^2} + \frac{\hat{B}_{22}}{(R_\beta)^2}$$

$$L_{37} = 2 \cdot A^* \cdot \overline{B}_{55} + \frac{A^* \cdot \overline{D}_{55}}{R_{\alpha\beta}} + \frac{D_{12} \cdot A^*}{R_\beta} + \frac{\overline{D}_{11} \cdot A^*}{R_\alpha},$$

$$L_{38} = -2 \cdot B^* \cdot \hat{B}_{44} - \frac{B^* \cdot \overline{D}_{44}}{R_\alpha} + \frac{D_{12} \cdot B^*}{R_\alpha} + \frac{\hat{D}_{22} \cdot B^*}{R_\beta}$$

$$L_{41} = L_{14},\ L_{42} = L_{24},\ L_{43} = L_{34}$$

$$L_{44} = -\overline{D}_{11} \cdot A^{*2} - \hat{D}_{66} \cdot B^{*2} - \overline{A}_{55} - \frac{\hat{D}_{44}}{(R_{\alpha\beta})^2},$$

$$L_{45} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*$$

$$L_{46} = B_{13} \cdot A^* + \frac{D_{12} \cdot A^*}{R_\beta} - \overline{B}_{55} \cdot A^* + \frac{\overline{D}_{11} \cdot A^*}{R_\alpha},$$

$$L_{47} = -\overline{E}_{11} \cdot A^* - \hat{E}_{66} \cdot B^{*2} - 2 \cdot \overline{B}_{55} + \frac{\overline{D}_{55}}{R_{\alpha\beta}} - \frac{\hat{E}_{44}}{(R_{\alpha\beta})^2}$$

$$L_{48} = -E_{12} \cdot A^* \cdot B^* - E_{66} \cdot A^* \cdot B^*,\ L_{51} = L_{15},\ L_{52} = L_{25},$$

$$L_{53} = L_{35},\ L_{54} = L_{45},\ L_{55} = -\hat{D}_{22} \cdot B^* - D_{66} \cdot A^{*2} - \hat{A}_{44} - \frac{\overline{D}_{44}}{(R_{\alpha\beta})^2},$$

$$L_{56} = \frac{D_{12} \cdot B^*}{R_\alpha} + \frac{\hat{D}_{22} \cdot B^*}{R_\beta} - \hat{B}_{44} \cdot B^*,\ L_{57} = -E_{12} \cdot A^* \cdot B^* - E_{66} \cdot A^* \cdot B^*,$$

$$L_{58} = -\hat{E}_{22} \cdot B^{*2} - \overline{E}_{66} \cdot A^{*2} - 2 \cdot \hat{B}_{44} - \frac{\hat{D}_{44}}{R_\beta} + \frac{\overline{E}_{44}}{R_\alpha \cdot R_{\alpha\beta}}$$

$$L_{61} = L_{16},\ L_{62} = L_{26},\ L_{63} = L_{36},\ L_{64} = L_{46},\ L_{65} = L_{56}$$

$$L_{66} = -\overline{D}_{55} \cdot A^{*2} - \hat{D}_{44} \cdot B^{*2} - A_{33} - \frac{B_{13}}{R_\alpha} - \frac{B_{23}}{R_\beta} - \frac{D_{11}}{(R_\alpha)^2} - \frac{\hat{D}_{22}}{(R_\beta)^2} - \frac{(\overline{B}_{36} + B_{36})}{R_{\alpha\beta}} - \frac{(\overline{D}_{66} + 2 \cdot D_{66} + \hat{D}_{66})}{(R_{\alpha\beta})^2}$$

$$L_{67} = -2 \cdot \overline{D}_{55} \cdot A^* + \frac{\overline{E}_{55} \cdot A^*}{R_{\alpha\beta}} + \frac{\overline{E}_{11} \cdot A^*}{R_\alpha} + \frac{E_{12} \cdot A^*}{R_\beta},$$

$$L_{68} = -2 \cdot \hat{D}_{55} \cdot B^* + \frac{\hat{E}_{44} \cdot B^*}{R_\beta} + \frac{E_{12} \cdot B^*}{R_\alpha} + \frac{\hat{E}_{22} \cdot B^*}{R_\beta},\ L_{71} = L_{17},\ L_{72} = L_{27},$$

$$L_{73} = L_{37},\ L_{74} = L_{47},\ L_{75} = L_{57},\ L_{76} = L_{67},\ L_{81} = L_{18},\ L_{84} = L_{48},\ L_{85} = L_{58}$$

$$L_{77} = -\overline{F}_{11} \cdot A^{*2} - \hat{F}_{66} \cdot B^{*2} - 4 \cdot \overline{D}_{55} + \frac{4 \cdot \overline{E}_{55}}{R_{\alpha\beta}} - \frac{\overline{F}_{55}}{(R_{\alpha\beta})^2}, \qquad (3.4)$$

$$L_{78} = -F_{12} \cdot A^* \cdot B^* - F_{66} \cdot A^* \cdot B^*,\ L_{82} = L_{28},\ L_{83} = L_{38},,\ L_{86} = L_{68}$$

$$L_{87} = L_{78},\ L_{88} = -\hat{F}_{22} \cdot B^{*2} - \overline{F}_{66} \cdot A^{*2} - \frac{\overline{F}_{66}}{(R_\alpha)^2} - \frac{\hat{E}_{44}}{R_\beta} - \frac{\hat{F}_{44}}{(R_\beta)^2} - 4 \cdot \hat{D}_{44}$$

The mass parameters M_{ij} is given by

$$M_{11} = \overline{I}_1,\ M_{14} = \overline{I}_2,\ M_{12} = M_{13} = M_{15} = M_{16} = M_{17} = M_{18} = 0$$

$$M_{22} = \overline{I}_1,\ M_{25} = \overline{I}_2,\ M_{21} = M_{23} = M_{24} = M_{26} = M_{27} = M_{28} = 0$$

$$M_{33} = \overline{I}_1,\ M_{31} = M_{32} = M_{34} = M_{35} = M_{36} = M_{37} = M_{38} = 0$$

$$M_{41} = \overline{I}_2,\ M_{44} = \overline{I}_3,\ M_{42} = M_{43} = M_{45} = M_{46} = M_{47} = M_{48} = 0$$

$$M_{52} = \overline{I}_2,\ M_{55} = \overline{I}_3,\ M_{51} = M_{53} = M_{54} = M_{56} = M_{57} = M_{58} = 0$$

$$M_{66} = \overline{I}_3,\ M_{61} = M_{62} = M_{63} = M_{64} = M_{65} = M_{67} = M_{68} = 0$$

$$M_{77} = \overline{I}_4,\ M_{71} = M_{72} = M_{73} = M_{74} = M_{75} = M_{76} = M_{78} = 0$$

$$M_{88} = \overline{I}_4,\ M_{82} = M_{81} = M_{83} = M_{84} = M_{85} = M_{86} = M_{87} = 0 \qquad (3.5)$$

The force vector $\{f_{ij}\}$ is given by:

$$q_\alpha = \sum_{k,l=1}^{N} q_{\alpha,kl}\ Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\quad q_\beta = \sum_{k,l=1}^{N} q_{\beta,kl}\ Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t}$$
$$q_n = \sum_{k,l=1}^{N} q_{n,kl}\ Sin(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\quad m_\alpha^{(1)} = \sum_{k,l=1}^{N} m_{\alpha,kl}^{(1)}\ Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t}$$
$$m_\beta^{(1)} = \sum_{k,l=1}^{N} m_{\beta,kl}^{(1)}\ Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t},\quad m_z = \sum_{k,l=1}^{N} m_{z,kl}\ Sin(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t} \qquad (3.6)$$
$$m_\alpha^{(2)} = \sum_{k,l=1}^{N} m_{\alpha,kl}^{(2)}\ Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\quad m_\beta^{(2)} = \sum_{k,l=1}^{N} m_{\beta,kl}^{(2)}\ Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t}$$

4. Numerical Results and Discussion

In the above section we have given the mathematical formulation of thick shell, and an extension of first order shear deformation shell theory. The additional new parameters using the third order shear deformation shell theory by Zannon (TSDTZ)obtained from Fourier expansion, which is given in the table 4.1 and compared with the first order shear deformation shell theory from the literature.

TABLE 4.1: Comparison of parameters of first order versus third order shear deformation shell theory

	First order SDST (Literature)	TSDTZ (Present study)
Condition	$\varepsilon_z = 0$	$\varepsilon_z \neq 0$
Unknowns Displacements/rotation	$u_0, v_0, w_0, \psi_\alpha, \psi_\beta$	$u_0, v_0, w_0, \psi_\alpha, \psi_\beta, \psi_z, \varphi_\alpha, \varphi_\beta$
Force resultants	$N_\alpha, N_\beta, N_{\alpha\beta}, N_{\beta\alpha}, Q_\alpha, Q_\beta$	$N_\alpha, N_\beta, N_{\alpha\beta}, N_z, N_{\beta\alpha}, Q_\alpha, Q_\beta$
Moment resultants	$M_\alpha, M_\beta, M_{\alpha\beta}, M_{\beta\alpha}, P_\alpha, P_\beta$	$M_\alpha^{(1)}, M_\alpha^{(2)}, M_\beta^{(1)}, M_\beta^{(2)}, M_{\alpha\beta}^{(1)}, M_{\alpha\beta}^{(2)}$, $M_{\beta\alpha}^{(1)}, M_{\beta\alpha}^{(2)}, P_\alpha^{(1)}, P_\alpha^{(2)}, P_\beta^{(1)}, P_\beta^{(2)}$
Strains at a point (3D)	$\varepsilon_{0\alpha}, \varepsilon_{0\beta}, \varepsilon_{0\alpha\beta}, \varepsilon_{0\beta\alpha}, \gamma_{0\alpha z}, \gamma_{0\beta z}$	$\kappa_\alpha^{(1)}, \kappa_\alpha^{(2)}, \kappa_\beta^{(1)}, \kappa_\beta^{(2)}$, $\kappa_{\alpha\beta}^{(1)}, \kappa_{\alpha\beta}^{(2)}, \kappa_{\beta\alpha}^{(1)}, \kappa_{\beta\alpha}^{(2)}$
Strains at the middle surface	$\kappa_\alpha, \kappa_\beta, \kappa_{\alpha\beta}, \kappa_{\beta\alpha}$	$\kappa_\alpha^{(1)}, \kappa_\alpha^{(2)}, \kappa_\beta^{(1)}, \kappa_\beta^{(2)}$, $\kappa_{\alpha\beta}^{(1)}, \kappa_{\alpha\beta}^{(2)}, \kappa_{\beta\alpha}^{(1)}, \kappa_{\beta\alpha}^{(2)}$
Equations		
Motion	5	8
Strain-displacement	10	15
Stress-strain	12	19

The above equation (3.2) in the matrix form is solved using MATLAB commercial code. This code is specifically modified for static vibration problems of simply supported cross-ply laminated composite shells. The solution is also valid for cylindrical shells having principle radii $R_\alpha = R_{\alpha\beta} = \infty$. For the numerical computation we compared the results with the orthotropic material properties of the cylindrical shells having length-to-arc ratio of one unit (i.e. $a/b = 1$), and the Poisson ratio of 0.25. The shear correction factors (K) for both directions are taken as 5/6. In static analyses, shells are under uniformly distributed load q. Thus, using a Fourier analysis, one finds the coefficients of a Fourier transform as $q_{mn} = 16q/mn\pi^2$ in Eqs. (3.3-3.4). Numerical investigation showed that the terms m and n did not need to exceed fifty for convergence of the results. Dimensionless transverse displacement, moment and force resultants

$$w^* = 10^3 E_2 h^3 w / q_n a^4$$
$$M_i^* = 10^3 M_i / qa^2,$$
$$N_i^* = 10^3 N_i / qa^3,$$

Where $i = \alpha, \beta,$ at the center of the shells are calculated based upon both FSDTQ and TSDTZ. Table 4.2 and 4.3 shows dimensionless displacement and force and moment resultants at the center of isotropic shells with different thickness ratios a/h

= 10, and 20 has been calculated for two-ply unsymmetrical [90/0] shells and three-ply symmetric [0/90/0] laminated orthotropic composite cylindrical shells for fixed thickness ratio (a/h) and various values of curvature (a/R) ratios by third order shear deformation theory. The results obtained by TSDTZ are then compared with earlier available results first order shear deformation theory by Qatu (FSDTQ), and three dimensional elasticity from finite element method (FEM). This supports us to evaluate the validity of the present TSDTZ theory.

TABLE 4.2.Comparison of dimensionless displacement and force and moment resultants of 2-ply unsymmetrical $[90/0]$ orthotropic cylindrical shells

$a/b = 1,\ K^2 = 5/6,\ E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{13} = G_{12},\ \nu_{12} = 0.25,\ a/h = 20$

a/R	Method	w^*	M_α^*	M_β^*	N_α^*	N_β^*
0.5	FSDTQ (Qatu et al., 2010, 2012)	11.632	52.953	28.801	1086.1	994.64
	TSDTZ (Present theory)	11.624	53.055	28.992	1085.2	993.78
	3D (Qatu et al., 2010, 2012)	11.612	53.689	29.643	1084.7	990.78
1	FSDTQ (Qatu et al., 2010, 2012)	5.6782	31.255	5.5480	1070.5	959.45
	TSDTZ (Present theory)	5.6741	31.341	5.5887	1070.5	959.87
	3D (Qatu et al., 2010, 2012)	5.7646	31.638	5.9669	1070.5	957.92

TABLE 4.3.Comparison of dimensionless displacement and force and moment resultants of 3-ply symmetric $[0/90/0]$ orthotropic cylindrical shells

$a/b = 1,\ K^2 = 5/6,\ E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{13} = G_{12},\ \nu_{12} = 0.25,\ a/h = 10$

a/R	Method	w^*	M_α^*	M_β^*	N_α^*	N_β^*
0.5	FSDTQ (Qatu et al., 2010, 2012)	9.5159	115.57	11.391	224.27	201.89

	TSDTZ (Present theory)	9.5851	114.005	11.594	226.02	202.32
	3D (Qatu et al., 2010, 2012)	10.661	112.55	13.847	249.78	222.52
	FSDTQ (Qatu et al., 2010, 2012)	7.8589	95.411	7.9335	370.82	335.59
1	TSDTZ (Present theory)	7.9409	93.229	8.0001	371.24	336.05
	3D (Qatu et al., 2010, 2012)	8.6399	91.340	9.5561	405.29	362.88

5. Conclusion

In this article we derived the mathematical formulation of a thick shell theory, which is an extension of the most popular first order deformation theory for shell structures. The present approximation of third order shear deformation theory Zannon (TSDTZ) by for simply supported cross-ply thick shell is developed using Fourier series expansion. Also, developed a simplified three dimension to two dimensional theories with a third order shear deformation theory for shells. In a forthcoming article we will use this solution to evaluate the results for free vibration analysis of the cross-ply thick shells and will assess the accuracy obtained from the numerical computation using the proposed theory.

6. References

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